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Gradient Flow for the Helfrich Variational Problem

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Abstract

The gradient flow associated to the Helfrich variational problem, called the *Helfrich flow*, is considered. A local existence result of n-dimensional Helfrich flow is given for any n.We also discuss known results,related topics,the development of our research group in this decade,and some open problems.

1 The Helfrich variational problem and its background

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed and oriented hypersurface immersed in $\mathbb{R}^{n+1}$. We do not assume that the inclusion $\Sigma \subset \mathbb{R}^{n+1}$ is an embedding. The function H stands for the mean curvature. The integral

$$\int_{\Sigma} H^2 dS$$

is called the *Willmore functional*, in which many mathematicians have been interested.

Now consider a variational problem for a functional related with the Willmore functional under some constraints. Let $A(\Sigma)$ be the area of Σ. The vectors $\boldsymbol{f}$ and $\boldsymbol{\nu}$ are the position vector of a point on Σ and the unit normal vector there respectively. Put

$$V(\Sigma) = -\frac{1}{n+1}\int_{\Sigma} \boldsymbol{f}\cdot\boldsymbol{\nu}\, dS.$$

This is the enclosed volume, when Σ is an embedded hypersurface and $\boldsymbol{\nu}$ is the inner normal. For given constants c_0, A_0, and V_0, consider critical points of

$$W(\Sigma) = \frac{n}{2} \int_{\Sigma} (H - c_0)^2 dS$$

under the constrains $A(\Sigma) = A_0$, $V(\Sigma) = V_0$.

This problem was firstly proposed by Helfrich [5] as a model of shape transformation theory of human red blood cells. For this case n is 2, and c_0 is the spontaneous curvature which is determined by the molecular structure of cell membrane. The surface Σ stands for the cell membrane.

For $n = 1$, the functional is

$$\frac{1}{2} \int_{\Sigma} \kappa^2 ds - c_0 \int_{\Sigma} \kappa \, ds + \frac{1}{2} c_0^2 \int_{\Sigma} ds,$$

where $\kappa (= H)$ is the curvature of the curve Σ, and s is the arch-length parameter. If we consider the variational problem under the constrain of length A among curves with fixed rotation number, then we can replace the functional with the first integral $\frac{1}{2} \int_{\Sigma} \kappa^2 ds$ only. Because the second and third integrals are respectively constant multiples of rotation number and the length, which are invariant in our problem. According to [3], a shape transformation of a closed loop of plastic tape between two parallel flat plates is governed by the one-dimensional Helfrich variational problem. This problem is also related with the spectral optimization problem for plain domains. Let Ω be a bounded plane domain, and Σ be its boundary. The function $G(x, y, t)$ is the Green function for the heat equation on $\Omega \times (0, T)$. The asymptotic expansion

$$\int_{\Omega} G(x, x, t) \, dx = \frac{1}{4\pi t} \left(a_0 + a_1 t^{\frac{1}{2}} + a_2 t + a_3 t^{\frac{3}{2}} + \cdots \right) \quad (t \to +0)$$

are well-known as the trace formula. Here

$$a_0 = V(\Sigma), \quad a_1 = -\frac{\sqrt{\pi}}{2} A(\Sigma), \quad a_2 = \frac{1}{3} \int_{\Sigma} \kappa \, ds \quad a_3 = \frac{\sqrt{\pi}}{64} \int_{\Sigma} \kappa^2 ds.$$

a_2 is determined by the topology of Ω. Hence the one-dimensional Helfrich problem is equivalent to the following problem: For given a_0, a_1 and a_2 find the domain Ω which minimize a_3. This problem was proposed and investigated by Watanabe [19, 20].

2 Known results

By the method of Lagrange multipliers, the Helfrich variational problem is described as

$$\delta W(\Sigma) + \lambda_1 \delta A(\Sigma) + \lambda_2 V(\Sigma) = 0.$$

Here δ stands for the first variation, and λ_j's are Lagrange multipliers. According to [4], the above equation becomes

$$\Delta_g H + (H - c_0)\left\{\frac{n^2}{2} H(H + c_0) + R\right\} - \lambda_1 n H - \lambda_2 = 0.$$

Here Δ_g is the Laplace-Beltrami operator, and R is the scalar curvature. Regarding Σ as the image $\boldsymbol{f}(\Sigma_0)$ of a $(n-1)$-dimensional manifold Σ_0, we obtain a quasilinear elliptic equation of forth order.

The two-dimensional Helfrich problem has a long history, and there are several known facts. It is easy to see spheres are critical points. In 1977, Jenkins [6] had found bifurcating solutions from spheres numerically. Subsequently Peterson [16] and Ou-Yang-Helfrich [15] formally investigated their stability/instability. Their arguments were justified mathematically by Takagi and the author in [11]. Au-Wan [2] considered critical points far from spheres but with rotational symmetry. Critical points without rotational symmetry were constructed by Takagi and the author [12].

In this article, we consider the associated gradient flow, called the *Helfrich flow*

$$v(t) = -\delta W(\Sigma(t)) - \lambda_1 \delta A(\Sigma(t)) - \lambda_2 \delta V(\Sigma(t)). \tag{2.1}$$

The function $v = \partial_t \boldsymbol{f} \cdot \boldsymbol{\nu}$ is the normal velocity of deformation of families of hypersurfaces $\Sigma(t)$. We shall overview known results about the Helfrich flow in the next section.

3 The Helfrich flow

In considering the flow problem, the multiplies are unknown functions of t. It is natural that they are determined so that $\frac{d}{dt}A(\Sigma(t)) = \frac{d}{dt}V(\Sigma(t)) = 0$. We have

$$\frac{d}{dt}A(\Sigma(t)) = \langle \delta A(\Sigma(t)), v(t)\rangle, \quad \frac{d}{dt}V(\Sigma(t)) = \langle \delta V(\Sigma(t)), v(t)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the $L^2(\Sigma(t)$-inner product. It follows from these and (2.1) that

$$\begin{aligned} &\begin{pmatrix} \langle \delta A(\Sigma(t)), \delta A(\Sigma(t))\rangle & \langle \delta V(\Sigma(t)), \delta A(\Sigma(t))\rangle \\ \langle \delta A(\Sigma(t)), \delta V(\Sigma(t))\rangle & \langle \delta V(\Sigma(t)), \delta V(\Sigma(t))\rangle \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ &\qquad\qquad = -\begin{pmatrix} \langle \delta A(\Sigma(t)), \delta W(\Sigma(t))\rangle \\ \langle \delta V(\Sigma(t)), \delta W(\Sigma(t))\rangle \end{pmatrix}. \end{aligned} \tag{3.1}$$

Denote the Gramian of the left-hand side by $G(\Sigma(t))$. If $G(\Sigma(t))$ does not vanish, then the multipliers are uniquely determined by the above relation. In this case we denote

$$\lambda_j = \lambda_j(\Sigma(t)).$$

When $G(\Sigma(t))$ vanishes, the multiplies are not uniquely determined, but we can show that $\lambda_1 \delta A(\Sigma(t)) + \lambda_2 \delta(\Sigma(t))$ is uniquely determined.

Theorem 3.1 *Let $P(\Sigma)$ be the orthogonal projection from $L^2(\Sigma)$ to $\left(\operatorname{span}_{L^2(\Sigma)}\{\delta A(\Sigma), \delta V(\Sigma)\}\right)^{\perp}$. Then the equation of Helfrich flow can be written as*

$$v(t) = -P(\Sigma(t))\delta W(\Sigma(t)) \quad (t > 0). \tag{3.2}$$

Solutions, if exist, satisfy

$$\frac{d}{dt}W(\Sigma(t)) \equiv -\|v(t)\|^2_{L^2(\Sigma(t))}, \quad \frac{d}{dt}A(\Sigma(t)) \equiv 0, \quad \frac{d}{dt}V(\Sigma(t)) \equiv 0. \tag{3.3}$$

We get the existence and uniqueness of the initial value problem. Let Σ_0 be the initial hypersurface, and h^α be the little Hölder space.

Theorem 3.2 (i) *Assume that Σ_0 is in the class of $h^{3+\alpha}$ for some $\alpha \in (0,1)$, and that $G(\Sigma_0) \neq 0$. Then there exists $T > 0$ such that there uniquely exists the solution $\{\Sigma(t)\}_{0 \leqq t < T}$ of (3.2) satisfying $\Sigma(0) = \Sigma_0$.*

(ii) *Assume that $G(\Sigma_0) = 0$. H_0 and R_0 are the mean curvature and the scalar curvature of Σ_0 respectively. Put*

$$\bar{H}_0 = \frac{1}{A(\Sigma_0)} \int_{\Sigma_0} H_0 dS, \quad \tilde{R}_0 = R_0 - \frac{1}{A(\Sigma_0)} \int_{\Sigma_0} R_0 dS.$$

If $\left(\bar{H}_0 - c_0\right)\tilde{R}_0 \equiv 0$, then there exists a global solution $\{\Sigma(t)\}_{t \geqq 0}$ of (3.2) satisfying $\Sigma(0) = \Sigma_0$.

Remark 3.1 The uniqueness is uncertain in the case (ii). We, however, can show the uniqueness when $n = 1$. See Theorem 5.1.

Sketches of proofs shall be given in the next two sections. For details, see [13].

The low-dimensional Helfrich flow has been considered in [7] (for $n = 2$) and in [9] (for $n = 1$).

In [7], the multiplier λ_j's are not determined as above, but are given as "known" constants. That is, for given $\{\lambda_1, \lambda_2, \Sigma_0\}$ as the data, solutions of (2.1) were constructed. Of course, solutions do not satisfy $\frac{d}{dt}A(\Sigma(t)) \equiv 0$,

$\frac{d}{dt}V(\Sigma(t)) \equiv 0$, and we cannot expect the global existence. Indeed, there exist solutions blowing up in finite/infinite time. The problem is shifted to find triples $\{\lambda_1, \lambda_2, \Sigma_0\}$ so that the solution can extend globally in time. In [7], the existence of such triples near spheres. Furthermore, such triples form a finite dimensional center manifold. The class of initial surfaces is $h^{2+\alpha}$ for some $\alpha \in (0,1)$, which is wider than ours. In our formulation $\nabla_g H$ appears in the concrete expression of $\lambda_j(\Sigma(t))$, and therefore we need extra regularity of Σ_0 than [7]. See Remark 5.1 below.

In [9], the gradient flow $\{\Sigma_\varepsilon(t)\}$ associated with the functional

$$W(\Sigma_\varepsilon) + \frac{1}{2\varepsilon}(A(\Sigma_\varepsilon) - A_0)^2 + \frac{1}{2\varepsilon}(V(\Sigma_\varepsilon) - V_0)^2$$

was constructed. The solution of (2.1) was obtained as the limit of $\{\Sigma_\varepsilon(t)\}$ as $\varepsilon \to +0$. This is a global solution, and satisfies (3.3). The class of initial curves is C^∞, but the uniqueness was uncertain.

4 Proof of Theorem 3.1

Theorem 3.1 is a special case of general theory of *projected gradient flow* [18].

We denote $\Sigma(t)$ simply by Σ. $\|\cdot\|$ stands for the $L^2(\Sigma)$-norm. Put

$$\tilde{H} = H - \frac{1}{A(\Sigma)}\int_\Sigma H\,dS, \quad H_* = \begin{cases} \frac{\tilde{H}}{\|\tilde{H}\|} & \text{if } \tilde{H} \not\equiv 0, \\ 0 & \text{if } \tilde{H} \equiv 0, \end{cases} \quad 1_* = \frac{1}{\|1\|}.$$

Note that $\langle H_*, 1_*\rangle = 0$. Since $\delta A(\Sigma) = -nH$ and $\delta V(\Sigma) = -1$, we have

$$\text{span}_{L^2(\Sigma)}\{\delta A(\Sigma), \delta V(\Sigma)\} = \text{span}_{L^2(\Sigma)}\{H, 1\} = \text{span}_{L^2(\Sigma)}\{H_*, 1_*\}.$$

Hence (2.1) becomes

$$v = -\delta W(\Sigma) - \lambda_1\delta A(\Sigma) - \lambda_2\delta V(\Sigma) = -\delta W(\Sigma) - \mu_1 1_* - \mu_2 H_* \qquad (4.1)$$

for some μ_j. It follows from $\frac{d}{dt}A(\Sigma) = \frac{d}{dt}V(\Sigma) = 0$ that

$$\langle 1_*, v\rangle = \langle H_*, v\rangle = 0.$$

Taking the $L^2(\Sigma)$-inner product (4.1) and 1_*, H_*, we get

$$0 = \langle 1_*, v\rangle = \langle 1_*, \delta W(\Sigma)\rangle - \mu_1, \quad 0 = \langle H_*, v\rangle = \langle H_*, \delta W(\Sigma)\rangle - \mu_2\|H_*\|^2.$$

In spite of $H_* \equiv 0$ or not, it holds that

$$-\mu_1 1_* - \mu_2 H_* = \langle 1_*, \delta W(\Sigma)\rangle 1_* + \langle H_*, \delta W(\Sigma)\rangle H_*.$$

Consequently we obtain (3.2).

It holds for solutions to (3.2) that

$$\begin{aligned}\frac{d}{dt}W(\Sigma) &= \langle \delta W(\Sigma), v\rangle = \langle \delta W(\Sigma), -P(\Sigma)\delta W(\Sigma)\rangle \\ &= -\|P(\Sigma)\delta W(\Sigma)\|^2 = -\|v\|^2.\end{aligned}$$

Since $v \in (\mathrm{span}\,\{\delta A(\Sigma), \delta V(\Sigma)\})^{\perp}$, we have

$$\frac{d}{dt}A(\Sigma) = \langle \delta A(\Sigma), v\rangle = 0, \quad \frac{d}{dt}V(\Sigma) = \langle \delta V(\Sigma), v\rangle = 0.$$

□

5 Sketch of Proof of Theorem 3.2

The local existence for the case $G(\Sigma_0) \neq 0$ is in a similar manner to [7]. If the Helfrich flow with $\Sigma(0) = \Sigma_0$ exists, and if Σ is close to Σ_0 in C^2-sense for small $t > 0$, then $G(\Sigma) \neq 0$. It follows from (3.1) that

$$\begin{pmatrix} \lambda_1(\Sigma) \\ \lambda_2(\Sigma) \end{pmatrix} = -\frac{1}{G(\Sigma)} \begin{pmatrix} \langle \delta V(\Sigma), \delta V(\Sigma) & -\langle \delta V(\Sigma), \delta A(\Sigma) \\ -\langle \delta A(\Sigma), \delta V(\Sigma) & \langle \delta A(\Sigma), \delta A(\Sigma) \end{pmatrix} \begin{pmatrix} \langle \delta A(\Sigma), \delta W(\Sigma) \\ \langle \delta V(\Sigma), \delta W(\Sigma) \end{pmatrix}. \tag{5.1}$$

Taking into the first variation formulas of A, V, and W (see [4]), we have

$$\langle \delta A(\Sigma), \delta A(\Sigma)\rangle = n^2 \int_\Sigma H^2 dS, \quad \langle \delta A(\Sigma), \delta V(\Sigma)\rangle = n \int_\Sigma H\, dS,$$

$$\langle \delta V(\Sigma), \delta V(\Sigma)\rangle = \int_\Sigma dS,$$

$$\langle \delta A(\Sigma), \delta W(\Sigma)\rangle = n \int_\Sigma \left(|\nabla_g H|^2 - \frac{1}{2}n^2 H^4 + H^2 R - c_0 H R + \frac{1}{2} n c_0^2 H^2 \right) dS,$$

$$\langle \delta V(\Sigma), \delta W(\Sigma)\rangle = \int_\Sigma \left(-\frac{1}{2}n^2 H^3 + HR - c_0 R + \frac{1}{2} n c_0^2 H \right) dS,$$

$$G(\Sigma) = \int_\Sigma n^2 H^2 dS \int_\Sigma dS - \left(\int_\Sigma nH\, dS \right)^2 = n^2 \int_\Sigma dS \int_\Sigma \tilde{H}^2 dS. \tag{5.2}$$

Inserting these into (5.1), we have the concrete expression of $\lambda_j(\Sigma)$'s. Thus we get

Proposition 5.1 *When $G(\Sigma) \neq 0$, the Lagrange multiplies $\lambda_j(\Sigma)$ are written by*

$$\int_\Sigma |\nabla_g H|^2 dS, \quad \int_\Sigma H^p dS \quad (p = 0,\, 1,\, 2,\, 3,\, 4), \quad \int_\Sigma H^q R\, dS \quad (q = 0,\, 1,\, 2),$$

on which the multipliers analytically depend.

In order to prove Theorem 3.2 (i), we regard Σ as the perturbation of Σ_0 in the normal direction with signed distance ρ. It is possible for a short time interval. Let $\bigcup_{\ell=1}^{m} U_\ell$ be an open covering of Σ_0. We denote the inner unit normal vector fields of Σ_0 by $\boldsymbol{\nu}_0$. The mapping $X_\ell : U_\ell \times (-a, a) \ni (\boldsymbol{s}, r) \to \boldsymbol{s} + r\boldsymbol{\nu}_0(\boldsymbol{s}) \in \mathbb{R}^{n+1}$ is a C^∞-diffeomorphism from $U_\ell \times (-a, a)$ to $\mathcal{R}_\ell = \mathrm{Im}(X_\ell)$ provided $a > 0$ is sufficiently small. Let denote the inverse mapping X_ℓ^{-1} by (S_ℓ, Λ_ℓ), where $S_\ell(X_\ell(\boldsymbol{s}, r)) = \boldsymbol{s} \in U_\ell$, and $\Lambda_\ell(X_\ell(\boldsymbol{s}, r)) = r \in (-a, a)$.

When $\Sigma(t)$ is sufficiently close to Σ_0 for small $t > 0$, we can represent it as a graph of a function on Σ_0 as

$$\Sigma_{\rho(t)} = \Sigma(t) = \bigcup_{\ell=1}^{m} \mathrm{Im}\left(X_\ell : U_\ell \to \mathbb{R}^{n+1},\ [\boldsymbol{s} \mapsto X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t))]\right).$$

Conversely for a given function $\rho : \Sigma_0 \times [0, T) \to (-a, a)$ we define the mapping $\Phi_{\ell,\rho}$ from $\mathcal{R}_\ell \times [0, T)$ to $\mathbb{R}$ by

$$\Phi_{\ell,\rho}(x, t) = \Lambda_\ell(x) - \rho(S_\ell(x), t).$$

Then $\left(\Phi_{\ell,\rho}(\cdot, t)\right)^{-1}(0)$ gives the surface $\Sigma_{\rho(t)}$.

The velocity in the direction of inner normal vector field of $\Sigma = \{\Sigma_{\rho(t)} \mid t \in [0, T)\}$ at $(x, t) = (X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t)), t)$ is given by

$$v(\boldsymbol{s}, t) = - \left.\frac{\partial_t \Phi_{\ell,\rho}(x, t)}{\|\nabla_x \Phi_{\ell,\rho}(x, t)\|}\right|_{x = X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t))} = \left.\frac{\partial_t \rho(\boldsymbol{s}, t)}{\|\nabla_x \Phi_{\ell,\rho}(x, t)\|}\right|_{x = X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t))}.$$

We can write down the Laplace-Beltrami operator, the mean curvature, the scalar curvature, and the Lagrange multipliers in terms of the function ρ and its derivatives, denoted $\mathit{\Delta}_\rho$, $H(\rho)$, $R(\rho)$, and $\lambda_j(\rho)$ respectively. Then the equation (3.2) is represented as

$$\begin{aligned} \partial_t \rho = L_\rho \Big(& -\mathit{\Delta}_\rho H(\rho) - \frac{1}{2} n^2 H^3(\rho) + H(\rho) R(\rho) - c_0 R(\rho) + \frac{1}{2} n c_0^2 H(\rho) \\ & + \lambda_1(\rho) n H(\rho) + \lambda_2(\rho) \Big), \end{aligned} \tag{5.3}$$

where

$$L_\rho = \|\nabla_x \Phi_{\ell,\rho}(x,t)\|\big|_{x=X_\ell(s,\rho(s,t))}.$$

We can find the expression of not only Δ_ρ, $H(\rho)$ but also the Gaussian curvature $K(\rho)$ in [7] for the case $n=2$. In our case the expression of Δ_ρ and $H(\rho)$ is the same as in [7], and we can get that of $R(\rho)$ in a similar way. In particular $\lambda_j(\rho)$ can be written in terms of ρ and its derivatives up to third order. Combining Proposition 5.1, we can see that the right-hand side of (5.3) is linear with respect to the fourth-order derivative of ρ, but not linear with respect to lower derivatives. The principal term $-L_\rho \Delta_\rho H(\rho)$ is the same as the equation dealt with [7, (2.1)]. Let $h^\gamma(\Sigma_0)$ be the little Hölder space on Σ_0 of order γ. We fix $0<\alpha<\beta<1$. Then, for $\beta_0 \in (\alpha,\beta)$ and $a>0$, put

$$\mathcal{U} = \{\rho \in h^{3+\beta_0}(\Sigma_0) \mid \|\rho\|_{C^2(\Sigma_0)} < a\}.$$

For two Banach spaces E_0 and E_1 satisfying $E_1 \hookrightarrow E_0$, the set $\mathcal{H}(E_1,E_0)$ is the class of $A \in \mathcal{L}(E_1,E_0)$ such that $-A$, considered as an unbounded operator in E_0, generates a strongly continuous analytic semigroup on E_0.

Proposition 5.2 *There exist*

$$Q \in C^\infty(\mathcal{U}, \mathcal{H}(h^{4+\alpha}(\Sigma_0), h^\alpha(\Sigma_0))), \quad F \in C^\infty(\mathcal{U}, h^{\beta_0}(\Sigma_0))$$

such that the equation (5.3) *is in the form*

$$\rho_t + Q(\rho)\rho + F(\rho) = 0.$$

Applying [1, Theorem 12.1] with $X_\beta = \mathcal{U}$, $E_1 = h^{4+\alpha}(\Sigma_0)$, $E_0 = h^\alpha(\Sigma_0)$, and $E_\gamma = h^{\beta_0}(\Sigma_0)$, we get the assertion (i) in Theorem 3.2.

Remark 5.1 The equation dealt with in [7] is a similar fourth-order equation, but linear with respect to the third order derivative of ρ. Therefore it was solvable for initial data in the class $h^{2+\alpha}$.

Now consider the assertion (ii) in Theorem 3.2. Before going to prove, we see an example of Σ_0 satisfying $G(\Sigma_0)=0$ and $\left(\bar{H}_0 - c_0\right)\tilde{R}_0 \equiv 0$. A typical example is a sphere. Indeed, spheres have constant mean curvature, and there for $G(\Sigma_0)=0$ (see (5.2)). Since the scalar curvature is also constant, we have $\tilde{R}_0 = 0$. Furthermore spheres are stationary solutions to (3.2).

To show the assertion (ii), it is enough to see that Σ_0 is a stationary solution.

Assume that $G(\Sigma)=0$. It follows from (5.2) that Σ has a constant mean curvature $H=\bar{H}$. Hence

$$\operatorname{span}_{L^2(\Sigma)}\{\delta A(\Sigma), \delta V(\Sigma)\} = \operatorname{span}_{L^2(\Sigma)}\{1\},$$

and

$$P(\Sigma)\phi = \phi - \frac{1}{A(\Sigma)} \int_\Sigma \phi \, dS$$

for $\phi \in L^2(\Sigma)$. Therefore at the time when $G(\Sigma(t)) = 0$, the equation (3.2) becomes

$$\begin{aligned} v(t) &= -\delta W(\Sigma) + \frac{1}{A(\Sigma)} \int_\Sigma \delta W(\Sigma) \, dS \\ &= -\Delta_g \bar{H} - \frac{1}{2} n^2 \bar{H}^3 + \bar{H} R - c_0 R + \frac{1}{2} n c_0^2 \bar{H} \\ &\quad + \frac{1}{A(\Sigma)} \int_\Sigma \left(\frac{1}{2} n^2 \bar{H}^3 - \bar{H} R + c_0 R - \frac{1}{2} n c_0^2 \bar{H} \right) dS \\ &= -\left(\bar{H} - c_0\right) \tilde{R}, \end{aligned}$$

where

$$\tilde{R} = R - \frac{1}{A(\Sigma)} \int_\Sigma R \, dS.$$

Consequently if the hypersurface Σ_0 satisfies $G(\Sigma_0) = 0$ and $\left(\bar{H}_0 - c_0\right) \tilde{R}_0 \equiv 0$, then it is a stationary solution of (3.2). □

We do not know the uniqueness in case of Theorem 3.2 (ii), expect for $n = 1$.

Theorem 5.1 *Consider the one-dimensional Helfrich flow. If Σ_0 satisfies $G(\Sigma_0) = 0$, then $\{\Sigma(t) \equiv \Sigma_0\}$ is the unique global solution with $\Sigma(0) = \Sigma_0$.*

Remark 5.2 When $n = 1$, the scalar curvature is zero by its definition, and therefore the condition $\left(\bar{H}_0 - c_0\right) \tilde{R}_0 \equiv 0$ is automatically satisfied.

Proof. When $n = 1$, the integral $\int_\Sigma H \, dS$ is a constant multiple of the rotation number. Therefore it does not depend on t. Consequently we have

$$\frac{d}{dt} G(\Sigma) = A_0 \frac{d}{dt} \int_\Sigma H^2 dS = 2A_0 \frac{d}{dt} W(\Sigma) = -2A_0 \|v\|^2 \leqq 0.$$

Combining this with $G(\Sigma) \geqq 0$ (see (5.2)), it holds that $G(\Sigma) \equiv 0$ provided $G(\Sigma_0) = 0$. Using the above relation again, we have $v \equiv 0$, that is, Σ is stationary. □

6 Gramian estimates

Assume that $G(\Sigma_0) \neq 0$, then we may do $G(\Sigma) \neq 0$ for small $t > 0$. Since $(G(\Sigma))^{-1}$ appears in the equation, it is desirable for proving global existence of solutions to have some a propri estimates of $G(\Sigma)$. It follows from (5.2) that $G(\Sigma) \geqq 0$, which is algebraically trivial since it is a Gramian. Now we consider lower bounds of G.

Proposition 6.1 *We have*

$$G(\Sigma) \geqq \frac{n^2 \left\{ A(\Sigma)^2 - (n+1)V(\Sigma) \int_\Sigma H \, dS \right\}^2}{A(\Sigma) \int_\Sigma \left(\tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu} \right)^2 dS},$$

where

$$\tilde{\boldsymbol{f}} = \boldsymbol{f} - \frac{1}{A(\Sigma)} \int_\Sigma \boldsymbol{f} \, dS.$$

Proof. It follows from $\delta A = -nH$, $\delta V = -1$ and scaling argument that

$$\langle \delta A, \tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu} \rangle = nA, \quad \langle \delta A, \tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu} \rangle = (n+1)V.$$

Therefore we obtain

$$\begin{aligned} n \left| A - (n+1)\bar{H}V \right| &= \left| \langle \delta A - n\bar{H}\delta V, \tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu} \rangle \right| \\ &= \left| \langle n\tilde{H}, \tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu} \rangle \right| \leqq n \|\tilde{H}\| \|\tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu}\| \end{aligned}$$

Combining (5.2), we get the assertion. □

This is an a priori lower bound of $G(\Sigma)$ when $n = 1$. To see this, putting $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \tilde{f}_2)$, we have

$$|\tilde{f}_i|^2 \leqq A(\Sigma) \int_\Sigma |\partial_s f_i|^2 ds = A(\Sigma) \int_\Sigma \tau_i^2 ds.$$

Summing up with respect to i, we get

$$\|\tilde{\boldsymbol{f}}\|_\infty \leqq A(\Sigma)$$

Therefore Proposition 6.1 implies

$$G(\Sigma) \geqq \left(1 - \frac{2V(\Sigma)}{A(\Sigma)^2} \int_\Sigma \kappa \, ds \right)^2.$$

Since $A(\Sigma)$, $V(\Sigma)$, and $\int_\Sigma \kappa\, ds$ are invariant, the estimate is a priori.

Let $n \geqq 2$, and let $L_1(\Sigma)$ be the first eigenvalue of $-\Delta_g$. Putting $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \cdots, \tilde{f}_n)$, we have

$$\begin{aligned}
\int_\Sigma \left(\tilde{\boldsymbol{f}} \cdot \boldsymbol{\nu}\right)^2 dS &\leqq \sum_i \int_\Sigma |\tilde{f}_i|^2 dS \\
&\leqq L_1^{-1}(\Sigma) \sum_i \int_\Sigma |\nabla f_i|^2 dS = L_1^{-1}(\Sigma) \sum_i \int_\Sigma g^{jk} \partial_j f_i \partial_k f_i dS \\
&= L_1^{-1}(\Sigma) \int_\Sigma g^{jk} \partial_j \boldsymbol{f} \cdot \partial_k \boldsymbol{f}\, dS = L_1^{-1}(\Sigma) \int_\Sigma g^{jk} g_{jk} dS \\
&= nA(\Sigma) L_1^{-1}(\Sigma).
\end{aligned}$$

Combining Proposition 6.1, we have a lower estimate of $G(\Sigma)$, but it is not a priori. Because $\int_\Sigma H\, dS$ and $L_1(\Sigma)$ may depend on t.

7 Related and open problems

Okabe [14] considered the gradient flow associated with

$$\int_\Sigma \kappa^2 ds$$

under constraints

$$A(\Sigma) = A_0, \quad \gamma(\Sigma) = 1.$$

Here γ is the local length defined as below. Let $\boldsymbol{f}(\theta)$ be a family of curves, where θ is a fixed coordinate. The local length is given by

$$\gamma = \|\partial_\theta f\|_{\mathbb{R}^2}.$$

It is a function on the curve, hence the corresponding multiplier is pointwise. Since γ depends on the choice of coordinate, it is not a geometrical quantity. Consequently there is a tangential component in the equation. For the gradient flow with one constraint

$$\gamma(\Sigma) = 1,$$

see [8]. For the comparison Okabe's result with the one-dimensional Helfrich flow, see [10].

In [9], the global existence of one-dimensional Helfrich flow, however, the global solvability of multi-dimensional Helfrich flow is still open. The asymptotic behavior has not been investigated yet.

In connection with the global existence, it is interesting to show a priori estimate of $G(\Sigma)$ for the case $n \geqq 2$, for example, an estimate in terms of $A(\Sigma)$, $V(\Sigma)$, and $\int_\Sigma K\, dS$, which are invariant. Here K is the Gauß curvature.

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